A Taste of Proofs

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1 The Pigeonhole Principle

The **Pigeonhole Principle** can be summarized by this analogy: If you have n pigeonholes (i.e. nests, I think?) and $n+1$ pigeons, then no matter how you put the pigeons into the pigeonholes, there will always be some pigeonhole that has at least two pigeons. This idea can be applied to many situations. For example, on a three-question true-or-false test, there will always be two questions with the same answer. If you roll a six-sided dice seven times, then there will always be two rolls that are the same number. If you ask 367 people for the month and day of their birthday, then there will always be two people with the same birthday.

The Pigeonhole Principle is something that mathematicians call an **axiom**, which is a statement that we can assume is true, because intuitively it should be obvious. But we like axioms because they are very useful when it comes to proving mathematical statements that are not so obvious. This is the basis of mathematical proofs: using what we take to be true to show that other statements are true. In this lecture, we will take a look at some proofs, and we will try to use Pigeonhole Principle to show why they are true.

1.1 Practice

1. Place five points inside an equilateral triangle with side length 1 (i.e. in the interior, and not on the edge). Prove that it is impossible to place the points such that the distance between any pair of points is more than 0.5.

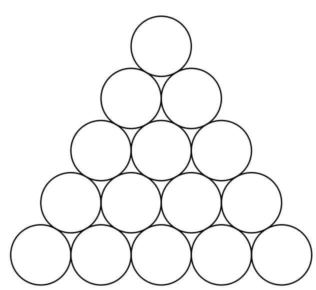
2. On a piano keyboard, the pitches of the notes are in this order:

   A, A#, B, C, C#, D, D#, E, F, F#, G, G#

   After the note G#, the notes after just repeat back to A, then A#, and so on. So in total, there are twelve distinct pitches that cycle repeatedly to fill up the whole keyboard.

   A tritone is a pair of two notes that are six spaces away from each other on the keyboard. For example, A and D# make a tritone, as well as B and F. Prove that if you pick any group of seven distinct pitches on the keyboard, there will be at least one tritone in the group.

3. You have fifteen billiard balls that are arranged in a triangle like so:

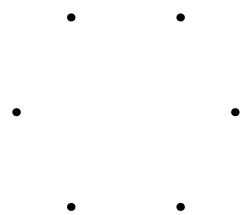
   Seven of these balls are classified as stripes, and eight of these balls are classified as solids. Prove that no matter how we arrange the stripes and solids in this triangle, there will always be two stripes that touch.

4. In a 3×4 rectangle, 6 points are placed. Prove that it is impossible to place the points such that the distance between any pair of points is more than $\sqrt{5}$.

   Hint: you will need to divide the rectangle into five pieces, where the longest distance between any pair of points within the same piece is no more than $\sqrt{5}$. It's harder than it looks!

1.2 Challenges

1. Draw six points on your paper like so:

   Choose two colors; let's say for example red and blue. Connect each pair of points with a red or a blue line segment, and you get to choose which color however you like. If done correctly, you will have drawn fifteen segments total. Prove that no matter how you pick the colors for your segments, you will always be able to find three of these points that form a triangle with all red or all blue sides.

2. Consider the fractions of the form $\frac{1}{n}$ for positive integers n. Some of these fractions can be written as terminating decimals, meaning that they stop at some point. For example, $\frac{1}{2} = 0.5$, $\frac{1}{5} = 0.2$, $\frac{1}{250} = 0.004$ are terminating. Others, such as $\frac{1}{3} = 0.3333\ldots$, can only be written as decimals that go on forever. However, there's a rule that because they're fractions, these decimals have repeating patterns, which is why they are called repeating decimals. For this problem, let's focus on the repeating decimals of the fractions $\frac{1}{n}$.

   The bar notation for fractions helps us rewrite the repeating decimals in a simpler way. We rewrite $0.3333\ldots$ as $0.\overline{3}$, signifying that all numbers under the line repeat forever. For example, $\frac{1}{6} = 0.1666\cdots = 0.1\overline{6}$ and $\frac{1}{11} = 0.090909\cdots = 0.\overline{09}$. By convention, we do not write $0.\overline{33}$ or $0.\overline{0909}$, since we always minimize the number of digits underneath the bar as much as possible.

   Sisyphus, who was condemned to eternal suffering in the underworld, decided that pushing a rock up a mountain was too much for him. He made a deal with Hades, who agreed to give him a chance to stop his suffering. Hades said, "find me a positive integer n such that the fraction $\frac{1}{n}$ is a repeating decimal that has at least n digits underneath the bar, and then you will no longer be punished." So Sisyphus, who isn't very clever, got to work trying to brute force the n that would satisfy Hades' needs. Many times, he got very close. He tried $\frac{1}{7}$ and got $0.\overline{142857}$, which has 6 digits underneath the bar. The value $n = 19$ is also close, as $\frac{1}{19} = 0.\overline{052631578947368421}$, which has 18 digits underneath the bar. He was very frustrated when he found out that the repeating decimal $\frac{1}{97}$ has 96 digits underneath the bar:

   $$0.\overline{010309278350515463917525773195876288659793814432989690721649484536082474226804123711340206185567}$$

   Now the question is, is this even possible? Is Sisyphus condemned to eternal psychological torture trying to find some value of n that satisfies Hades' conditions?

2 Invariants

Another possible strategy we can use to solve proofs is an invariant. In simple terms, an invariant is something that we notice will always happen no matter what we do. It is a sort of intermediate observation that we can make about the problem, which we can use as a stepping stone to solve the whole proof. For example, consider this problem:

2.1 Example of an Invariant

Take a standard 8×8 chessboard, but remove two opposite corner squares. You also have 31 dominoes, and every time you put down a domino, it will cover two adjacent squares of the board. Prove that it is impossible to completely cover the entire chessboard with these dominoes.

Solution: First we need to find an invariant, which is an observation that should always be true about the dominoes when we place them. One thing you might see is that each domino always covers a black square and a white square. This is true no matter how you place the domino. This is our invariant: each domino always covers one white and one black square. Then, using some thinking, we can use this observation to prove our original statement. Notice how on a chessboard, the opposite corner squares are the same color. This means that when we remove two opposite corner squares, we have an unequal amount of white and black squares. Because each domino covers one white and one black, it would only be possible to cover the board with dominoes if there was an equal number of white and black squares. Because that is not the case for our board, it is therefore impossible.

2.2 Practice

1. A knight in chess moves in a diagonalish way. To be specific, if a knight starts on the square S, it can finish its move on any of the eight squares labeled F, as shown below:

	F		F	
F				F
		S		
F				F
	F		F	

   Suppose a knight makes a series of moves that takes it right back to where it started. Prove that the knight must have made an even number of moves.

2. In the plane, there are three hockey pucks A, B, and C. A hockey player hits one of the pucks so that it passes through the segment of the other two pucks and stops at some point. For example, if the player hits A, then the puck must pass through the line segment BC before stopping. After 25 hits, is it possible that each of the pucks returns to its original position?

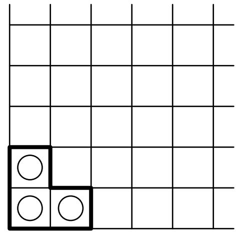
3. Prove it is impossible to cover a 10×10 chessboard with 25 T-shaped figures that are each 4 squares big.

2.3 Challenges

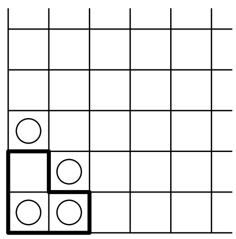
1. Suppose chameleons are either red, green, or blue, and whenever two of them of different colors meet, they both change to the third color. Otherwise, they don't change colors. Suppose you start with 12 red chameleons, 13 green chameleons, and 14 blue chameleons. Is it possible that at some point all the chameleons will be red?

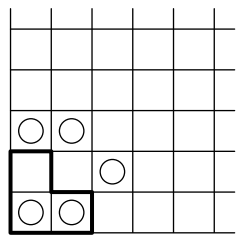
2. A 10×10 square field is divided into 100 equal square patches, 9 of which are overgrown with weeds. It is known that during a year the weeds spread to those patches that have no less than two neighboring (i.e., having a common side) patches that are already overgrown with weeds. Otherwise, weeds will not grow on that patch. Prove that the field will never overgrow completely with weeds.

2.4 Three Prisoners Problem (HARD)



Consider a grid that extends infinitely in the rightward and upward directions. At the start, three dots or "prisoners" are in the three cells in the bottom-left corner. These prisoners want to escape the prison (i.e. the outlined region), and they can move in the following way: if both the cell above and the cell to the right are empty, the prisoner can divide into two and move to those cells. Basically, remove the existing prisoner and fill in the cells above it and to the right of it. The two diagrams below show a possible sequence of two moves you can start with:





Is it possible for all prisoners to escape the prison? That is, is it possible for the three cells at the bottom-left to be empty?